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THE EFFECTS OF SMALL NOISE ON IMPLICITLY DEFINED  
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Deputy County Clerk

Approved by the Board of Supervisors and the Board of County Officers

Witness my hand and the seal of said County at San Diego, California

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Deputy County Clerk

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\* Research supported in part by the Air Force Office of Scientific Research under grant AFOSR-82-0133. I would like to thank William D. Kistner, R. Landman, S. Witter, L. Walker, and T. Whitt for helpful discussions.

**88 08 08 165** **UNCLASSIFIED**

THE EFFECT OF SMALL NOISE ON IMPLICITLY DEFINED  
NON-LINEAR DYNAMICAL SYSTEMS

Abstract

The dynamics of a large class of non-linear systems are described implicitly, i.e. as a combination of algebraic and differential equations. These dynamics admit of jump behavior. We extend the deterministic theory to a stochastic theory since (i) the deterministic theory is restrictive, (ii) the macroscopic deterministic description of dynamics frequently arises from an aggregation of microscopically fluctuating dynamics and (iii) to robustify the deterministic theory. We compare the stochastic theory with the deterministic one in the limit that the intensity of the additive white noise tends to zero. We study the modelling issues involved in applying this stochastic theory to the study of the noise behavior of a multivibrator circuit. Discuss the limitations of our methodology for certain classes of systems and present a modified approach for the analysis of sample functions of noisy non-linear circuits.

Keywords: Bifurcation, Singular Perturbation, Jump Behavior, Laplace's method, Noise behavior of non-linear circuits.

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Chief, Technical Information Division

## Section 1. Introduction

The dynamics of a large class of engineering systems are described only implicitly, for instance, those of non-linear circuits, swing dynamics of an interconnected power system, as also thermodynamic systems far from equilibrium. The implicit definition of their dynamics is as follows: the state variables are constrained to satisfy some algebraic equations, i.e. they are constrained to lie on a manifold  $M$  in the state space. The dynamics on this manifold  $M$  are then specified implicitly by specifying only the projection of the vector field on  $M$  onto a certain base space above which  $M$  lies. (i.e. a subspace of the original state space of the same dimension as  $M$ ). The process of obtaining the system dynamics explicitly consists of 'lifting' the specified velocities onto a vector field on  $M$  (lifting is the inverse of projecting). Lifting may not, however, be possible at points where the projection map (restricted to the tangent space of the constraint manifold) has singularities. This singularity is typically resolved by regularization, i.e. by interpreting the algebraic constraint equations as the singularly perturbed limit of 'parasitic' or fast dynamics. The dynamics of the original system are obtained as the degenerate limit of the dynamics of the regularized system - the resulting trajectories may be discontinuous and this is referred to as jump behavior.

The foregoing deterministic theory needs to be extended to a stochastic theory for three reasons:

- a) The conditions under which the limit trajectories to the regularizations exist are extremely restrictive so as to exclude several systems of interest.

b) Frequently, the algebraic constraint equations arise from the macroscopic aggregation of microscopically fluctuating dynamics, e.g. the flow of current in a resistor, the demand for electrical power at a distribution point in an electrical power network. More generally, deterministic equations describing thermodynamic systems are of this kind. Thus, the algebraic constraint equations contain in addition a rapidly fluctuating (or white noise) component.

c) The methods of analysis for deterministic systems of the implicitly defined kind involve techniques of bifurcation theory - their conclusions are extremely sensitive to imperfections and the addition of white noise.

Since in all the situations of interest to us, the intensity of the additive noise is small, we study in this paper the dynamics of implicitly defined dynamics in the presence of small additive noise. In fact, we compare the conclusions of the stochastic theory with those of the deterministic theory in the limit that the noise intensity tends to zero. The foregoing process requires the computation of two sets of limits: the limit that the regularization tends to zero and the limit that the intensity of the additive white noise tends to zero. In general, these limits do not commute. We explore in this paper the modelling issue of which sequence of limits is appropriate in the context of a specific system. The layout of the paper where we carry out this program is as follows:

In Section 2, we review briefly the dynamics of deterministic constrained systems and their jump behavior. With some minor modifications we follow here our earlier work [11] and the references contained therein.

In Section 3, we begin the study of noisy constrained dynamical systems. For the initial study we use as tools the work of Papanicolaou, et al [10] on martingale approaches to limit theorems. To study the



dynamics of noisy constrained systems in the presence of small noise, we develop and use in our context Laplace's method of steepest descent. We study in several separate cases, the comparison between the deterministic and small noise theory, describing: (i) how the stochastic theory yields conclusions about system dynamics when the deterministic theory fails and (ii) how the jump behavior of systems is modified by the presence of small noise. This section is a considerable extension of our previous work in the context of phase transitions in van der Waals gases [12]. Several examples are presented to instantiate our results.

In Section 4, we present the detailed deterministic analysis of Section 2 applied to the dynamics of an emitter coupled relaxation oscillator circuit. We then show that the experimental conclusions of Abidi [1] on the dynamics of these circuits in the presence of small noise seem not to agree with the stochastic theory presented in Section 3.

In Section 5, we discuss the sequences of limits implied by the development of Section 3 - and the nature of systems for which this development yields the correct conclusions. In particular, we show that the development of Section 3 is relevant to systems where the separation in time scales between the slow and fast components is very large and is more important than the small intensity of the white noise (characterized by a certain sequence of limits subsumed by Section 3) - for instance in phase transitions, reaction rates and other phenomena of non-equilibrium thermodynamics. For non-linear circuits, however, the separation in time scales is less marked, so that we present here the relevant analysis (sample function calculations) for these systems with the order of limits reversed from that of Section 3. We use as tools the foundational



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The intuitive picture that now emerges in the original  $t$ -timescale is as follows: For a hyperbolic equilibrium point  $(x_0, y_0)$  of the sped-up system  $S$  attach its stable manifold  $S_{y_0}^{x_0}$  transversally to  $M$ . When the attached manifold  $S_{y_0}^{x_0}$  is of dimension  $m$ , then disturbances and noise will not cause the "state"  $(x, y)$  of the system (2.1), (2.2) to be repelled from  $M$ . If, in fact, the attached manifold is of dimension  $< m$ , disturbances may cause the "state"  $(x, y)$  to be repelled from  $M$  and follow instantaneously the dynamics of the sped-up system  $S$  to a new  $\omega$ -limit set of (2.8). By assuming that (2.1) has only finitely many equilibrium points as its

set of  $\omega$ -limits, etc. for each  $x$ , we may guarantee that  $(x, y)$  will transiently visit  $(x, y)$  which is a new equilibrium point of  $S$ . If the new equilibrium point  $(x, y)$  is unstable it is likely that the system will further evolve till a new stable equilibrium point is reached.

With this, the intuitive picture of the dynamics with respect to the original system (2.1), (2.2) is as follows: If the system (2.1), (2.2) is in a state  $(x, y)$  which is a new equilibrium point of  $S$ , then

$$\dot{x} = 0, \quad \dot{y} = 0.$$

It follows that the system (2.1), (2.2) is in a state of equilibrium.

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Further, we assume that the only non-hyperbolic equilibrium points of the system  $S$  are those which have eigenvalues at the origin, rather than on the  $y$ -axis. (i.e. the only non-hyperbolic points are singular

points). This means the emergence of closed orbits in the Hopf bifurcation, and the emergence of limit cycles in the period-doubling bifurcation.

Assumption 2 (the  $\omega$ -limit set of  $S$ ): Several possibilities are possible for the  $\omega$ -limit set of  $S$ . Several possibilities are possible for the  $\omega$ -limit set of  $S$ .

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THEORY OF THE EARTH AND ITS HISTORY

The earth is a sphere, and its surface is divided into two main parts, the land and the water. The land is divided into continents and islands, and the water is divided into oceans and seas. The earth is covered by a thin layer of air, and the air is divided into layers. The earth is also covered by a thin layer of water, and the water is divided into layers.

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### 3.1 Controlled Dynamical Systems in the Presence of Drift-Driving Noise

Clearly, the Model (3.1) of fully controlled systems differs from the deterministic Model (2.1) in that a small  $\epsilon > 0$  in the random term, yielding (3.1) in equations (3.1) and (3.2) respectively, are small. In the present context, as we shall show in the next section,  $\epsilon$  and  $\sigma$  are of order of  $A^{-1/2}$  where  $A$  is the Boltzmann constant and  $T$  the temperature in degrees Kelvin, a quantity that is small at room temperatures. Thus, we compare the behavior of fully controlled dynamical systems with that of the deterministic controlled systems of Section 2 in the limit that  $\epsilon, \sigma$  the realizations of the driving noise go to zero. The Markov process and the Laplace method of steepest descent for asymptotic calculations (see e.g. Chap. 4 of [14] or the technique of Rajaguru et al. [15]) and the results of Ventcel-Freedman [16] (the semi-discrete case) several stages (leading) in the form of [17].

### 3.2 The Case of a Uniform Controlled Process

Assume that (3.1) is the gradient with respect to  $x$  of a function  $S(x, y)$ , i.e.,

$$\dot{x}(t) = -\frac{1}{\epsilon} \nabla_x S(x, y)$$

for some function  $S: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ . Then, the results of Iasnokolau, et al. [18] yield that provided the derivatives of  $S$  with respect to  $y$  grow rapidly enough at  $y$ , the density of the diffusion generated by (3.1) converges exponentially to

$$\bar{p}(x, y) = \bar{p}(x) \cdot \exp \left\{ -\frac{S(x, y)}{\epsilon} \right\} \quad (3.10)$$

where  $\bar{p}(x)$  is chosen such that

$$\int_{\mathbb{R}^n} \bar{p}(x, y) dx = 1$$

Note that for all  $\epsilon > 0$  and  $x \in \mathbb{R}^n$  the critical points (with respect to  $y$ ) of  $\bar{p}^{-1}(x, y)$  are the equilibrium points of the deterministic system (2.4) with  $x$  frozen given in this instance by

$$\dot{y} = -\frac{1}{2} \text{grad}_y S(x, y) \quad (3.11)$$

Further, if for some  $x_0$ ,  $S(x_0, y)$  is a Morse function (of  $y$ ), then for all  $\epsilon > 0$  every local maximum of  $\bar{p}^{-1}(x, y)$  is a stable equilibrium of (3.11).

To compare the noisy constrained system with the deterministic constrained system in the limit that  $\epsilon \rightarrow 0$ , it will be necessary to evaluate integrals like (3.9) in the limit that  $\epsilon \rightarrow 0$ . This is done using the following version of Laplace's method:

Theorem 3.2 (Laplace's Method)

Let for each  $x \in \mathbb{R}^n$ ,  $S(x, y)$  have global minima at  $y_1^*(x), y_2^*(x), \dots, y_N^*(x)$ , where  $N$  may depend on  $x$ . Let them all be non-degenerate. Further, let  $S(x, y)$  have at least quadratic growth (in  $y$ ) as  $y \rightarrow \infty$ . Then, in the limit that  $\epsilon \rightarrow 0$ ,  $\bar{p}^{-1}(x, y)$  converges to

$$\sum_{i=1}^N a_i(x) \delta(y - y_i^*(x)) / \sum_{i=1}^N a_i(x) \quad (3.12)$$

where  $a_i(x) = \det(D_y^2 S(x, y_i^*(x)))^{-1/2}$

More precisely, if  $\delta(x, y)$  is a smooth function having polynomial growth as  $y \rightarrow \infty$ , then

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \bar{p}^{-1}(x) &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^m} \delta(x, y) \bar{p}^{-1}(x, y) dy \\ &= \sum_{i=1}^N a_i(x) \delta(x, y_i^*(x)) / \sum_{i=1}^N a_i(x) \\ &= \bar{p}_0^{-1}(x) \end{aligned} \quad (3.13)$$

<sup>1</sup> i.e. the Hessian  $D_y^2 S(x, y)$  at  $y = y_i^*(x)$  is nonsingular

Proof: Since  $\bar{p}^\lambda(x, y) = \exp - \frac{S(x, y)}{\lambda} / \int_{\mathbb{R}^m} \exp - \frac{S(x, y)}{\lambda} dy$ ,

we will first evaluate

$$\int_{\mathbb{R}^m} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} dy$$

for simplicity first assume that  $S(x, y)$  has a single global minimum at  $\hat{y}$ .

We will then show that

$$\int_{\mathbb{R}^m} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} dy = \phi(x, y^*) \frac{(2\pi\lambda)^{m/2} \exp - S(x, y^*)/\lambda [1 + o(1)]}{[\det D_y^2 S(x, y^*)]^{1/2}} \quad (3.14)$$

First, by the Morse Lemma (see for e.g. Milnor [16]) there exists a neighborhood  $U$  of  $\hat{y}$  and a change of coordinates  $\mathbb{R}^m \rightarrow U$  given by  $y = \gamma(\bar{y})$  such that  $y^* = \gamma(0)$  and

$$S(x, y) = S(x, y^*) + \frac{1}{2} \sum_{i=1}^m (\bar{y}_i)^2 \quad (3.15)$$

Further, outside the neighborhood  $U$  of  $y^*$ ,  $S(x, y) > S(x, y^*) + \delta$

for some  $\delta > 0$  so that

$$\int_{\mathbb{R}^m/U} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} dy = \exp\left[-\frac{S(x, y^*)}{\lambda}\right] o(\lambda^{-1/2}) \quad (3.16)$$

for all  $\lambda > 0$ . Clearly, then (3.16) does not contribute to the leading term of (3.14). Consider now

$$\begin{aligned} & \int_U \phi(x, y) \exp - \frac{S(x, y)}{\lambda} dy \\ &= \exp - \frac{S(x, y^*)}{\lambda} \int_{\mathbb{R}^m} \exp \left( - \sum_{i=1}^m \frac{\bar{y}_i^2}{2\lambda} \right) \phi(x, \gamma(\bar{y})) |\det D \gamma(\bar{y})| d\bar{y} \end{aligned} \quad (3.17)$$

Now, standard manipulations with Gaussian distributions yield that

$$\int_{\mathbb{R}^m} (\exp - \frac{1}{2\lambda} \sum_{i=1}^m y_i^{-2}) \phi(\bar{y}) d\bar{y}$$

$$= (2\pi\lambda)^{m/2} (\phi(0) + o(\lambda)) \quad (3.18)$$

Thus, to evaluate (3.17) we only need compute  $|\det D \gamma(0)|$ . Differentiating (3.15) twice with respect to  $y$  yields

$$D_2^2 S(x, y) = ((\frac{d\bar{y}}{dy})^{-1})^T (\frac{d\bar{y}}{dy})^{-1} \quad (3.19)$$

From (3.19) it follows that

$$|\det D \gamma(0)| = [\det D_2^2 S(x, y^*)]^{-1/2}$$

so that (3.14) now is immediate on combining (3.16), (3.17) and (3.18)

In the instance that  $S(x, y)$  has several global minima  $y_1^*(x), y_2^*(x), \dots, y_N^*(x)$  it follows from an easy extension of the foregoing argument that

$$\int_{\mathbb{R}^m} \phi(x, y) \exp - \frac{S(x, y)}{\lambda} dy = (2\pi)^{m/2} \exp \frac{-S(x, y^*)}{\lambda} \left[ \sum_{i=1}^N \det [D_2^2 S(x, y_i^*(x))] \right]^{-1/2}$$

$$\phi(x, y_1^*(x)) + o(1) \quad (3.20)$$

Setting  $\phi(x, y) = 1$  in (3.20) yields the corresponding expression for

$$\int_{\mathbb{R}^m} \exp \frac{-S(x, y)}{\lambda} dy. \text{ Combining this with (3.20) we have equation (3.13).}$$

□

Remarks: (1) If the growth conditions on  $S(x, y)$  and  $\phi(x, y)$  are uniform in  $x$  for  $|x| \leq R$  it can be shown that for  $p \geq 1$

$$\int_{|x| \leq R} |\bar{f}_\lambda(x) - \bar{f}_0(x)|^p dx \rightarrow 0 \text{ as } \lambda \rightarrow 0 \quad (3.21)$$

Proof: Is presented in Sastry-Hijab [12].

Remark: The order of the limits is peculiar in Theorem (3.3). If the order is interchanged i.e.  $\lambda \downarrow 0$  first and then  $\varepsilon \downarrow 0$  it is clear that one recovers in the limit the deterministic development of Section 2 (with the minor modification that  $\dot{x}$  has an additive white noise terms. The jump-behavior of the y-variable is as explained in that section. If, however,  $\varepsilon \downarrow 0$  first and then  $\lambda \downarrow 0$ , the jump-behavior of the y-variable is somewhat different, as we now elaborate:

The behavior of the conditional density of y given x as  $\lambda \downarrow 0$  is as in Theorem 3.2: the y variable is at one of the global minima of  $S(x, \cdot)$  with probability proportional to the curvature of  $S(x, \cdot)$   $((\text{Det } D^2_x S(x, y_1^*))^{-1/2})$  at that minimum. Consider first the case when the minimum is unique. There is then a jump in the y-variable if there is a change in the global minimum of  $S(x, \cdot)$  as x is varied. Points of jump then will be points of appearance and disappearance of global minima of  $S(x, \cdot)$ . This is in contrast to the deterministic picture of Section 2, where, for the instance that  $g(x, y)$  is of the form of (3.11), stable equilibrium of the sped-up system S are local minima of  $S(x, \cdot)$  and points of bifurcation are points appearance and disappearance of local minima of  $S(x, \cdot)$ .

We illustrate this with an example - the van der Pol oscillator of (2.5), (2.6) with added noise. Consider

$$\begin{aligned}\dot{x} &= y + \sqrt{\varepsilon} \xi(t) \\ \varepsilon \dot{y} &= -x - y^3 + y + \sqrt{\varepsilon} \eta(t)\end{aligned}$$

Here  $S(x, y) = -xy - \frac{y^4}{4} + \frac{y^2}{2}$  so that, in the limit that  $\varepsilon \downarrow 0$ ; the x-process converges to one satisfying

The proof of (3.21) uses the dominated convergence and Spitzer's theorem

(2) If  $S(x, y)$  has a manifold  $M$  of global minima, then certainly these global minima cannot be non-degenerate. However, if  $S(x, y)$  is non-degenerate in directions orthogonal to  $M$ , then a minor modification of the preceding theorem yields

$$\int_{\mathbb{R}^n} \exp \left( -\frac{S(x, y)}{\epsilon} \right) dx dy \approx \frac{(2\pi)^{n/2}}{\epsilon^{n/2}} \int_M \frac{1}{\det D_x S(x, y)} dx dy$$

Where  $y$  is any point belonging to  $M$ ,  $\det D_x S(x, y)$  is the determinant of the non-degenerate part of the Hessian and  $dx$  is the Lebesgue measure on  $M$ .

We can now combine the results of Theorems 3.1 and 3.2, using a minor modification of the techniques of Karatzas and Karatzas.

Theorem 3.3 Weak convergence of  $(X_t^N)_{t \in [0, T]}$  as  $N \rightarrow \infty$ .

Given any  $T > 0$ , in the limit that  $N \rightarrow \infty$  and  $\epsilon_N \rightarrow 0$  fast enough, the first component  $x = x(t)$  of the solution  $(x, y)$  converges weakly in  $C([0, T]; \mathbb{R}^n)$  to the unique diffusion  $x = x(t)$  satisfying the law

$$\dot{x} = \bar{F}_0(x) + \sigma(x) \dot{W} \quad (3.4)$$

where

$$\bar{F}_0(x) = \frac{\sum_{i=1}^N a_i(x) f(x, y_i^*(x))}{\sum_{i=1}^N a_i(x)} \quad (3.5)$$

and the  $y_1^*(x), \dots, y_N^*(x)$  are the non-degenerate global minima of  $S(x, y)$

and  $a_i(x) = [\det D_x^2 S(x, y_i^*(x))]^{-1/2}$ .



# THE INTEGRAL OF A FUNCTION OF A COMPLEX VARIABLE

§ 1. INTRODUCTION

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

Let  $f(z)$  be a function of a complex variable  $z$  defined in a domain  $D$  of the complex plane. Let  $\gamma$  be a curve in  $D$  from a point  $a$  to a point  $b$ . The integral of  $f(z)$  along  $\gamma$  is defined by the formula

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

where  $\gamma(t)$  is a parametrization of the curve  $\gamma$  from  $a$  to  $b$ . The integral is independent of the parametrization of the curve  $\gamma$ .

## § 2. BASIC PROPERTIES

Let  $f(z)$  and  $g(z)$  be functions of a complex variable  $z$  defined in a domain  $D$ . Let  $\gamma$  be a curve in  $D$  from a point  $a$  to a point  $b$ . The integral of  $f(z) + g(z)$  along  $\gamma$  is equal to the sum of the integrals of  $f(z)$  and  $g(z)$  along  $\gamma$ .

$$\int_{\gamma} (f(z) + g(z)) dz = \int_{\gamma} f(z) dz + \int_{\gamma} g(z) dz$$

The integral of  $cf(z)$  along  $\gamma$  is equal to  $c$  times the integral of  $f(z)$  along  $\gamma$ .

$$\int_{\gamma} cf(z) dz = c \int_{\gamma} f(z) dz$$

$$\int_{\gamma} f(z) dz = - \int_{\gamma^{-1}} f(z) dz$$

Let  $\gamma_1$  and  $\gamma_2$  be two curves in  $D$  from a point  $a$  to a point  $b$ . The integral of  $f(z)$  along  $\gamma_1 + \gamma_2$  is equal to the sum of the integrals of  $f(z)$  along  $\gamma_1$  and  $\gamma_2$ .

$$\int_{\gamma_1 + \gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz$$

## § 3. THE INTEGRAL OF A FUNCTION OF A COMPLEX VARIABLE

Let  $f(z)$  be a function of a complex variable  $z$  defined in a domain  $D$ .

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

Let  $\gamma$  be a curve in  $D$  from a point  $a$  to a point  $b$ . The integral of  $f(z)$  along  $\gamma$  is defined by the formula

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

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1. The first part of the report is a general introduction to the subject of the study. It discusses the importance of the study and the objectives of the research.



Figure 1. A scatter plot showing data points and two clusters of points.

The second part of the report is a detailed description of the methodology used in the study. It outlines the experimental design, the data collection process, and the statistical analysis techniques employed.

The third part of the report presents the results of the study. It includes a series of tables and figures that illustrate the findings of the research.

The fourth part of the report is a discussion of the results and their implications. It compares the findings with previous research and discusses the potential applications of the study.

The fifth part of the report is a conclusion and a list of references.

The sixth part of the report is a list of references.

The seventh part of the report is a list of references.



### Section 4 The Effects of Thermal Noise on an Emitter-Coupled Relaxation Oscillator

We study in this section the tolerance of the theory developed in Sections 1 and 2 to the study of the effects of thermal noise on a relaxation oscillator. Figure 9 shows a simplified circuit diagram of such an oscillator. The Analog Devices AD 517. We discuss first using the terminology of Section 2 the deterministic description of the oscillator. The circuit equations are given by

$$\frac{1}{C} \frac{dV}{dt} = -\frac{1}{R} V + \frac{1}{R} V_{BE} \quad (4.1)$$

$$V_{BE} = V_{BE0} + V_T \ln \frac{I_C}{I_{S0}} \quad (4.2)$$

$$I_C = I_{C0} + I_T \ln \frac{V_{BE}}{V_{BE0}} \quad (4.3)$$

$$I_{C0} = I_{S0} \exp \left( \frac{V_{BE0}}{V_T} \right) \quad (4.4)$$

$$I_{S0} = I_{S1} \exp \left( \frac{V_{BE0}}{V_T} \right) \quad (4.5)$$

Here  $V_{BE0} = V_{BE}^*$  is the threshold voltage for the base emitter junction and  $I_{S0}$  is the reverse saturation current. The transistors are assumed to be identical. With  $V_{BE} = V_{BE0} + V_T \ln \frac{I_C}{I_{S0}}$ , we may combine equations (4.1) - (4.5) to obtain

$$\frac{1}{R} \frac{dV}{dt} = -\frac{1}{R} V + \frac{1}{R} V_{BE0} + \frac{1}{R} V_T \ln \frac{I_C}{I_{S0}} \quad (4.6)$$

$$I_C = I_{S0} \exp \left( \frac{V_{BE0}}{V_T} \right) \exp \left( \frac{V - V_{BE0}}{V_T} \right) \quad (4.7)$$

Equations (4.6), (4.7) form an implicitly defined dynamical system. The solution curve to the algebraic equation (4.7) is plotted in the  $(V, I_C)$



plane in Figure 10. Some of the features of this curve are noted below:

- (i) For  $-2I_0R < V < 2I_0R$  the equation (4.7) has three solutions, while for  $V = 2I_0R$  and  $V < 2I_0R$  the equation has only one solution.
- (ii) As  $V \rightarrow \infty$ ,  $i = 2I_0$  and as  $V \rightarrow -\infty$ ,  $i = 0$  asymptotically.
- (iii) The values  $V = 2I_0R$ ,  $i = \frac{V_T}{2R}$  and  $V = -2I_0R$ ,  $i = 2I_0 - \frac{V_T}{2R}$  are the points of bifurcation of equation (4.7) with  $V$  treated as the bifurcation parameter, i.e. at these points it is not possible to solve (4.7) for  $i$  as a function of  $V$  locally and uniquely. These points may be shown to be points of fold bifurcation.

Returning now to the full system - (4.6) and (4.7) we see that continuous solutions for the system exist so long as  $i$  can be solved continuously as a function of  $v$  in (4.7) so as to obtain:

$$\frac{dv}{dt} = \frac{V_0 - v}{\tau} \quad (4.6)$$

and

$$\frac{di}{dt} = \frac{V_0 - v}{\tau} \cdot \frac{V_0 - v}{V_0 - v + V_T + 2I_0(2I_0 - 1)i} \quad (4.8)$$

When  $-2I_0R + V_T + 2I_0(2I_0 - 1)i = 0$ , i.e.  $i = \frac{V_T}{2R}$  or  $i = 2I_0 - \frac{V_T}{2R}$  it appears that  $\frac{di}{dt}$  is infinite so as to prevent the integration of equations (4.6), (4.8). The regularization of this system is accomplished by taking into account the fact that parasitic capacitances present in the transistors, as well as the finite slew rate of the operational amplifiers will prevent  $i$  from varying discontinuously and in effect change the description of the

of the circuit dynamics from (4.6), (4.7) to

$$\frac{dV}{dt} = \frac{(I_0 - i)}{C} \quad (4.6)$$

$$i \frac{di}{dt} = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i \quad (4.9)$$

Equations (4.6) and (4.9) are a gross simplification of all the actual parasitics present in the circuit. A more detailed and exhaustive description involving all the parasitics would start from the original equations (4.1) - (4.5). The present regularized model is, however, accurate enough for our purposes. The phase portrait of this system shown in Figure 1 includes a single unstable equilibrium point ( $V=0$ ,  $i=I_0$ ) and a limit cycle. The limit trajectories of (4.6), (4.9) as  $\epsilon \rightarrow 0$  exist and include the relaxation oscillation shown in Figure 12 - a limit cycle with two discontinuities - at the points where the trajectory switches from the Q1 on, Q2 off 'state' to the Q1 off, Q2 on 'state' and vice versa. Note also from Figure 11 that the Q1 on, Q2 on 'state' is unstable as evidenced by the trajectories of (4.6), (4.9) pointing away from that 'state'. The current waveform  $i(t)$  is as shown in Figure 13. The half period of the oscillation  $T$  may be estimated approximately by integrating equation 4.8 with the approximation that for  $0 \leq t \leq T$ ,  $i \ll I_0$ , so that we have

$$T \approx \frac{C}{I_0} \int_{I_1}^{V_T/2R} \left( -2R + \frac{V_T}{i} \right) di$$

or

$$T \approx \frac{C}{I_0} \left[ 2R(-V_T/2R + I_1) + V_T \ln(V_T/2I_1R) \right] \quad (4.10)$$

From equation (4.10) it follows that the frequency of oscillation is (approximately) linearly proportional to  $I_0$ , which enables this oscillation to be used as an electronically tunable oscillator (e.g. in a phase locked loop). In such applications, it is important to know the noise characteristics of the oscillator in response to resistive thermal noise. Experimental observations of Abidi [1] indicate that the actual (noisy) current waveform is as shown in Figure 14. Key features of this figure are as follows:

- (a) the transitions or jumps appear to be noise free
- (b) the noise superimposed on the deterministic waveform of Figure 13 appears to be small (low intensity) immediately following a jump and then appear to build in intensity.

We assume (see e.g. [14]) that all the noise sources in the circuit can be lumped into a single-noisy current source  $i_n(t)$  shown dotted in Figure 9:  $i_n(t)$  is assumed to be white with intensity  $\lambda$  (with  $\lambda$  small at room temperatures, since it is proportional to  $kT$ ). It is easy to check that the equation (4.6) is now unchanged, while (4.7) changes to

$$0 = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i + 2R\sqrt{\lambda} i_n(t) \quad (4.11)$$

We regularize the system (4.6), (4.11) as before to obtain

$$\dot{V} = (I_0 - i)/C \quad (4.6)$$

$$\varepsilon \dot{i} = V - (2I_0 - 2i)R - V_T \ln(2I_0 - i)/i + 2R\sqrt{\varepsilon\lambda} i_n(t) \quad (4.12)$$

Note that  $\varepsilon$  scales the intensity of the white noise in (4.12) precisely for the same reason as in equation (3.2) of Section 3. The techniques of Section 3.2 may now be used to obtain that as  $\varepsilon \downarrow 0$ , the  $V$ -process converges weakly on  $C([0, T]; \mathbb{R})$  to one satisfying :

$$\dot{V} = (I_0 - \bar{i}^\lambda(V))/C$$

where  $\bar{i}^\lambda(V)$  is  $i$  integrated over the conditional density for  $i$  given  $V$ , in the limit that  $\epsilon \rightarrow 0$ ,  $\bar{p}^\lambda(i, V)$ . As in the example of Section 3.2, we have in the limit that  $\lambda \rightarrow 0$ ,  $\bar{p}^\lambda(i, V)$  converging to a sequence of delta functions jumping from one leg of the solution curve to (4.7) to the other at  $V=0$ . Also, choosing the interval of weak convergence to be large it appears that the relaxation oscillation is broken up.

This analysis is contrary to the experimental evidence of Abidi [1] What has gone wrong? How does one recover the experimental results of Abidi [1]? These are the questions that we taken up next.

## Section 5. Sample Function Calculations.

The mathematical reason for the anomaly between the machinery developed in Section 3 and the experimental conclusions of Section 4, is the order of limits  $\epsilon \downarrow 0$  followed by  $\lambda \downarrow 0$  in Theorem (3.3). This order of taking limits is suitable for explaining phenomena in several situations in non-equilibrium thermodynamics (for e.g. phase transitions of the kind discussed in Sastry-Hijab [12], Eyring chemical reaction rates, etc. - see for e.g. Nicolis-Prigogine [9], Landauer [6]). In fact, it has been noted by thermodynamicists of the Brussels School that "fluctuations play a crucial role in changing the behavior of systems near bifurcation fronts". However, this order of limits is not fully satisfactory in the circuit context. The reason for this lies in the fact that the order of limits  $\epsilon \downarrow 0$  followed by  $\lambda \downarrow 0$  (Theorem 3.3) yields the correct conclusions only when the dynamics of the fast (sped-up) system are much faster than those of the slower x variable. This is so, because, as we state in Section 3.4, Laplace's method of steepest descent picks for the limit values of  $\bar{p}^\lambda(x,y)$  as  $\lambda \downarrow 0$  the most stable  $\lambda$ -limit sets of the underlying deterministic systems. This in turn is consistent with the intuition that in the presence of persistent random perturbation (wide-band in nature) the trajectories of a system will concentrate after sufficiently long periods of time in the vicinity of the most-stable sets. However, the sufficiently long periods of time may be very large indeed. It is possible to show, for example in the gradient case of Section 3.2 that the average time required to escape from a stable equilibrium is of the order of  $e^{k/\lambda}$  for some  $k > 0$  (see for eg. Schuss [15], Ventsel-Freidlin [15]).

By taking limits in the sequence  $\epsilon \downarrow 0$  followed by  $\lambda \downarrow 0$ , the implication is that  $\epsilon$  is smaller than  $e^{-k/\lambda}$ , i.e.  $\epsilon$  is at least  $(e^{-k/\lambda})$ , so that the fast system has sufficiently much time to concentrate in the vicinity of its  $\lambda$ -limit sets. This is frequently the situation in non-equilibrium thermodynamics where the slower dynamics are frequently assumed to be 'quasi-static'. In the circuit context, however, the separation of time scales between the slow and fast variable is not as large as is implied by the theorem.

As noted in the remark following Theorem 3.3, if the order of limits is interchanged (i.e.  $\lambda \downarrow 0$  and then  $\epsilon \downarrow 0$ ), one recovers the deterministic development of Section 2. Before, we further elaborate and make precise the statements of the previous paragraph we indicate how one analyses sample functions of the process generated by (3.1), (3.2) in the limit that  $\lambda \downarrow 0$  followed by  $\epsilon \downarrow 0$ . The major tool in this development is the work of Wentzell-Freidlin [13].

We consider here sample functions of the process generated by

$$\dot{x} = f(x,y) + \sqrt{\epsilon} \dot{W}, \quad x(0) = x_0 \quad (5.1)$$

$$\dot{y} = g(x,y) + \sqrt{\epsilon} \dot{V}, \quad y(0) = y_0 \quad (5.2)$$

with precisely the same assumptions as in Section 3. Let  $\gamma = [\gamma_x, \gamma_y]: [0,T] \rightarrow \mathbb{R}^n \times \mathbb{R}^m$  be a  $C^1$  map from the interval  $[0,T]$  to the  $x, y$  space with  $\gamma_x(0) = x_0, \gamma_y(0) = y_0$ . Define, for this trajectory, the functional  $I_\epsilon(\gamma)$  by

$$I_\epsilon(\gamma) = \int_0^T \left\| \begin{bmatrix} \dot{\gamma}_x(t) - f(\gamma_x, \gamma_y) \\ \epsilon [\dot{\gamma}_y(t) - \frac{1}{\epsilon} g(\gamma_x, \gamma_y)] \end{bmatrix} \right\|^2 dt \quad (5.3)$$

Then, we have the following theorem for measuring the deviation of the sample functions of (5.1) from the stationary specified trajectory.

Theorem 5.1.1.1

For any  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$P\left\{\sup_{0 \leq t \leq T} |x(t) - \bar{x}(t)| < \delta \mid x(0) = x_0\right\} > 1 - \epsilon$$

Remark. (1) Equation (5.1) gives an estimate of how close the sample functions of the process of (5.1) come to the stationary specified trajectory  $\bar{x}(t) = \bar{P}^T x \bar{P}^T$ . (2) The estimate (5.1) is sharpest when the right hand side is as close to 0 as possible, i.e., when  $\epsilon$  is chosen to minimize (5.1). We examine this next next.

(2) Consider the definition of  $I_{\epsilon}$  in equation (5.1). Note that  $I_{\epsilon}(0, T) = \bar{P}^T x_0 \bar{P}^T$ . In fact, the solution to the  $n$  of the deterministic system starting from  $x_0$  will then  $I_{\epsilon} = 0$  is minimum value. Further,  $I_{\epsilon}$  is, for arbitrary  $\epsilon$ , a measure of how far  $x(t)$  fails to satisfy the  $\epsilon$  w.r.t. the deterministic system.

(3) Note the  $\epsilon$  weighting in the  $y$ -component of equation (5.1). Taking the limit that  $\epsilon \rightarrow 0$ , we see that so long as  $\dot{x}_y$  remains bounded the contribution of the second term is merely  $\frac{1}{2} \int_0^T \dot{x}_y^2 dt$ .

(4) Note that Theorem 5.1 gives an estimate for the deviation of the process of (5.1), (5.2) from the trajectory  $\bar{x}$  for fixed  $\epsilon$  and  $\delta > 0$  as follows: For  $\epsilon > 0$  with probability arbitrarily close to 1, the most likely trajectory for the process is that which minimizes  $I_{\epsilon} = 0$ . Further, taking the limit that





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Figure 1. The effect of the concentration of the *Agrobacterium* suspension on the transformation efficiency of *Agrobacterium* strains. The concentration of the *Agrobacterium* suspension was 10<sup>6</sup> cells/ml (○), 10<sup>7</sup> cells/ml (□), 10<sup>8</sup> cells/ml (△), 10<sup>9</sup> cells/ml (◇), and 10<sup>10</sup> cells/ml (●). The error bars represent the standard deviation of three independent experiments.

1. The first group of variables includes the demographic characteristics of the respondents, such as age, gender, and education level. These variables are used to control for potential confounding factors that may influence the dependent variable.

1. *Journal of the American Medical Association*, 1997; 277: 1033-1038.

1. *Journal of the American Medical Association*, 1997; 278: 1039-1044.

1. *Journal of the American Medical Association*, 1997; 277: 1033-1038.

1. *Journal of the American Medical Association*, 1997; 277: 1033-1036.

<sup>a</sup>  $\chi^2 = 0.76$ ,  $p = .82$ . <sup>b</sup>  $\chi^2 = 0.92$ ,  $p = .63$ . <sup>c</sup>  $\chi^2 = 0.00$ ,  $p = 1.00$ . <sup>d</sup>  $\chi^2 = 0.00$ ,  $p = 1.00$ . <sup>e</sup>  $\chi^2 = 0.00$ ,  $p = 1.00$ . <sup>f</sup>  $\chi^2 = 0.00$ ,  $p = 1.00$ .

*Journal of Management Education* 36(8) 907-924

1. *Chlorophyll a* (Chl *a*) and *Chlorophyll b* (Chl *b*) were determined using the method of Arar and Collins (1987). The concentration of Chl *a* and Chl *b* was expressed as  $\mu\text{g mL}^{-1}$  of the sample.

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1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020 2021 2022 2023 2024 2025 2026 2027 2028 2029 2030 2031 2032 2033 2034 2035 2036 2037 2038 2039 2040 2041 2042 2043 2044 2045 2046 2047 2048 2049 2050 2051 2052 2053 2054 2055 2056 2057 2058 2059 2060 2061 2062 2063 2064 2065 2066 2067 2068 2069 2070 2071 2072 2073 2074 2075 2076 2077 2078 2079 2080 2081 2082 2083 2084 2085 2086 2087 2088 2089 2090 2091 2092 2093 2094 2095 2096 2097 2098 2099 2100 2101 2102 2103 2104 2105 2106 2107 2108 2109 2110 2111 2112 2113 2114 2115 2116 2117 2118 2119 2120 2121 2122 2123 2124 2125 2126 2127 2128 2129 2130 2131 2132 2133 2134 2135 2136 2137 2138 2139 2140 2141 2142 2143 2144 2145 2146 2147 2148 2149 2150 2151 2152 2153 2154 2155 2156 2157 2158 2159 2160 2161 2162 2163 2164 2165 2166 2167 2168 2169 2170 2171 2172 2173 2174 2175 2176 2177 2178 2179 2180 2181 2182 2183 2184 2185 2186 2187 2188 2189 2190 2191 2192 2193 2194 2195 2196 2197 2198 2199 2200 2201 2202 2203 2204 2205 2206 2207 2208 2209 2210 2211 2212 2213 2214 2215 2216 2217 2218 2219 2220 2221 2222 2223 2224 2225 2226 2227 2228 2229 2230 2231 2232 2233 2234 2235 2236 2237 2238 2239 2240 2241 2242 2243 2244 2245 2246 2247 2248 2249 2250 2251 2252 2253 2254 2255 2256 2257 2258 2259 2260 2261 2262 2263 2264 2265 2266 2267 2268 2269 2270 2271 2272 2273 2274 2275 2276 2277 2278 2279 2280 2281 2282 2283 2284 2285 2286 2287 2288 2289 2290 2291 2292 2293 2294 2295 2296 2297 2298 2299 2300 2301 2302 2303 2304 2305 2306 2307 2308 2309 2310 2311 2312 2313 2314 2315 2316 2317 2318 2319 2320 2321 2322 2323 2324 2325 2326 2327 2328 2329 2330 2331 2332 2333 2334 2335 2336 2337 2338 2339 2340 2341 2342 2343 2344 2345 2346 2347 2348 2349 2350 2351 2352 2353 2354 2355 2356 2357 2358 2359 2360 2361 2362 2363 2364 2365 2366 2367 2368 2369 2370 2371 2372 2373 2374 2375 2376 2377 2378 2379 2380 2381 2382 2383 2384 2385 2386 2387 2388 2389 2390 2391 2392 2393 2394 2395 2396 2397 2398 2399 2400 2401 2402 2403 2404 2405 2406 2407 2408 2409 2410 2411 2412 2413 2414 2415 2416 2417 2418 2419 2420 2421 2422 2423 2424 2425 2426 2427 2428 2429 2430 2431 2432 2433 2434 2435 2436 2437 2438 2439 2440 2441 2442 2443 2444 2445 2446 2447 2448 2449 2450 2451 2452 2453 2454 2455 2456 2457 2458 2459 2460 2461 2462 2463 2464 2465 2466 2467 2468 2469 2470 2471 2472 2473 2474 2475 2476 2477 2478 2479 2480 2481 2482 2483 2484 2485 2486 2487 2488 2489 2490 2491 2492 2493 2494 2495 2496 2497 2498 2499 2500 2501 2502 2503 2504 2505 2506 2507 2508 2509 2510 2511 2512 2513 2514 2515 2516 2517 2518 2519 2520 2521 2522 2523 2524 2525 2526 2527 2528 2529 2530 2531 2532 2533 2534 2535 2536 2537 2538 2539 2540 2541 2542 2543 2544 2545 2546 2547 2548 2549 2550 2551 2552 2553 2554 2555 2556 2557 2558 2559 2560 2561 2562 2563 2564 2565 2566 2567 2568 2569 2570 2571 2572 2573 2574 2575 2576 2577 2578 2579 2580 2581 2582 2583 2584 2585 2586 2587 2588 2589 2590 2591 2592 2593 2594 2595 2596 2597 2598 2599 2600 2601 2602 2603 2604 2605 2606 2607 2608 2609 2610 2611 2612 2613 2614 2615 2616 2617 2618 2619 2620 2621 2622 2623 2624 2625 2626 2627 2628 2629 2630 2631 2632 2633 2634 2635 2636 2637 2638 2639 2640 2641 2642 2643 2644 2645 2646 2647 2648 2649 2650 2651 2652 2653 2654 2655 2656 2657 2658 2659 2660 2661 2662 2663 2664 2665 2666 2667 2668 2669 2670 2671 2672 2673 2674 2675 2676 2677 2678 2679 2680 2681 2682 2683 2684 2685 2686 2687 2688 2689 2690 2691 2692 2693 2694 2695 2696 2697 2698 2699 2700 2701 2702 2703 2704 2705 2706 2707 2708 2709 2710 2711 2712 2713 2714 2715 2716 2717 2718 2719 2720 2721 2722 2723 2724 2725 2726 2727 2728 2729 2730 2731 2732 2733 2734 2735 2736 2737 2738 2739 2740 2741 2742 2743 2744 2745 2746 2747 2748 2749 2750 2751 2752 2753 2754 2755 2756 2757 2758 2759 2760 2761 2762 2763 2764 2765 2766 2767 2768 2769 2770 2771 2772 2773 2774 2775 2776 2777 2778 2779 2780 2781 2782 2783 2784 2785 2786 2787 2788 2789 2790 2791 2792 2793 2794 2795 2796 2797 2798 2799 2800 2801 2802 2803 2804 2805 2806 2807 2808 2809 2810



APPENDIX

1. General Information - This section contains information regarding the project, including the title, author, and date of completion.
2. Objectives - This section outlines the goals and objectives of the project, as well as the scope of the study.
3. Methodology - This section describes the methods and procedures used in the study, including data collection and analysis techniques.
4. Results - This section presents the findings of the study, including data tables, graphs, and charts.
5. Conclusions - This section summarizes the main findings of the study and provides recommendations for future research.
6. References - This section lists the sources of information used in the study, including books, articles, and websites.
7. Appendix - This section contains supplementary information, including raw data, additional tables, and figures.
8. Tables - This section contains a list of tables included in the study, along with their titles and descriptions.
9. Figures - This section contains a list of figures included in the study, along with their titles and descriptions.
10. Index - This section provides a list of keywords and terms used in the study, along with their corresponding page numbers.

1. The first of the two main lines of the drawing is a straight line, the second is a curve.

2. The first of the two main lines of the drawing is a straight line, the second is a curve.

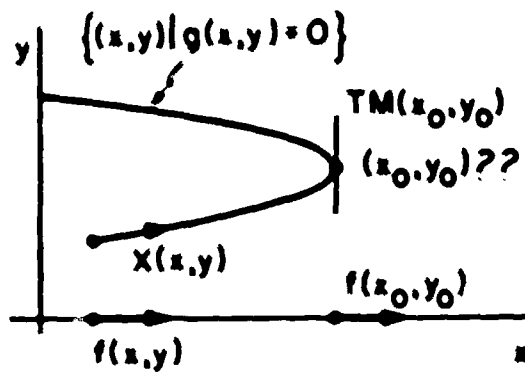


Figure 1: Illustration of the left value of the function  $f(x, y)$  for  $(x, y) \in \{(x, y) | g(x, y) = 0\}$ .

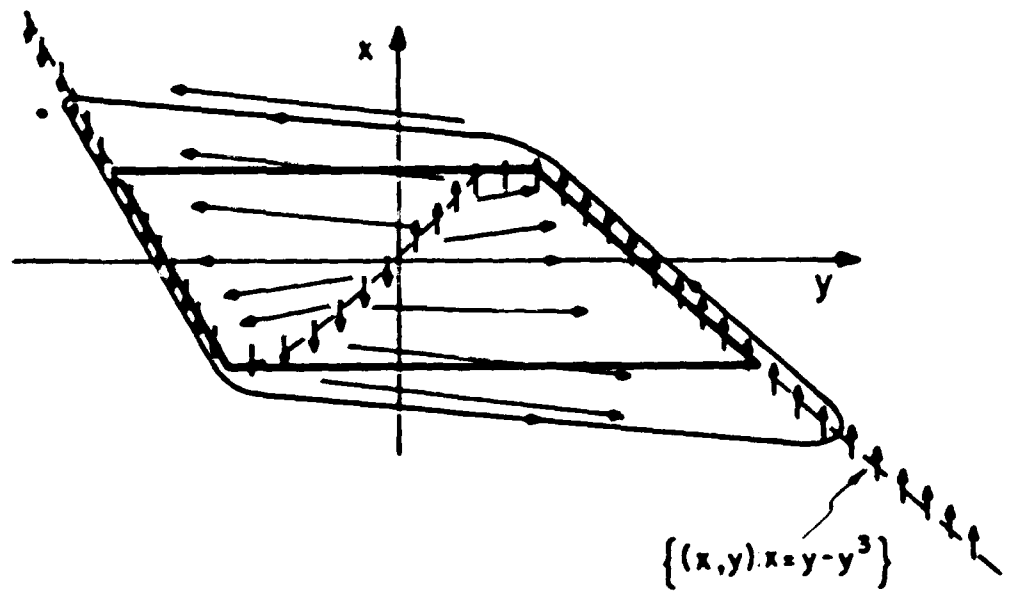


Figure 2: Showing the dynamics of the degenerate and regularized van der Pol oscillator.

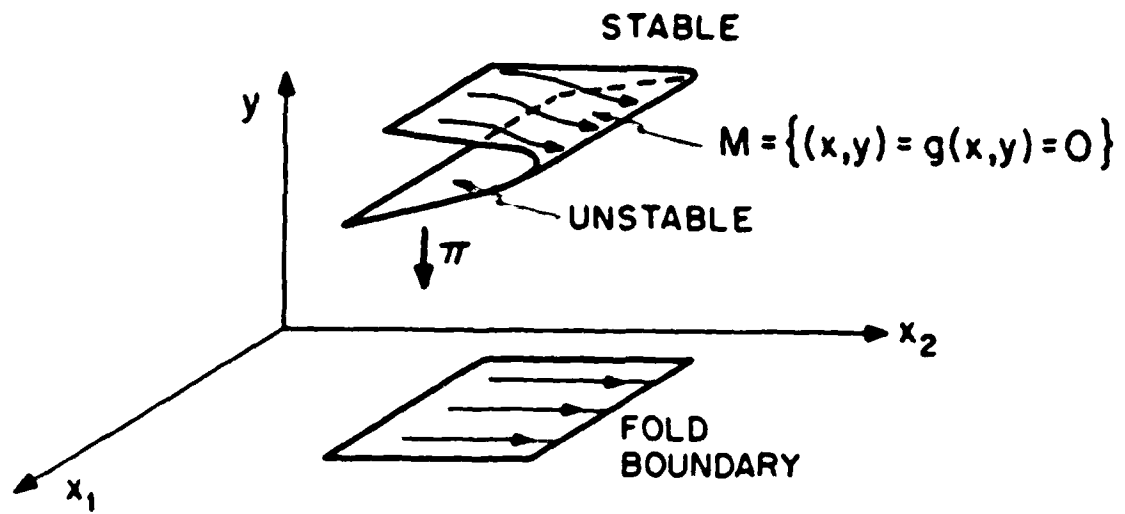


Figure 3: Visualization of a Fold Bifurcation and the Trajectories in  $M$  that lead towards the Fold Boundary.

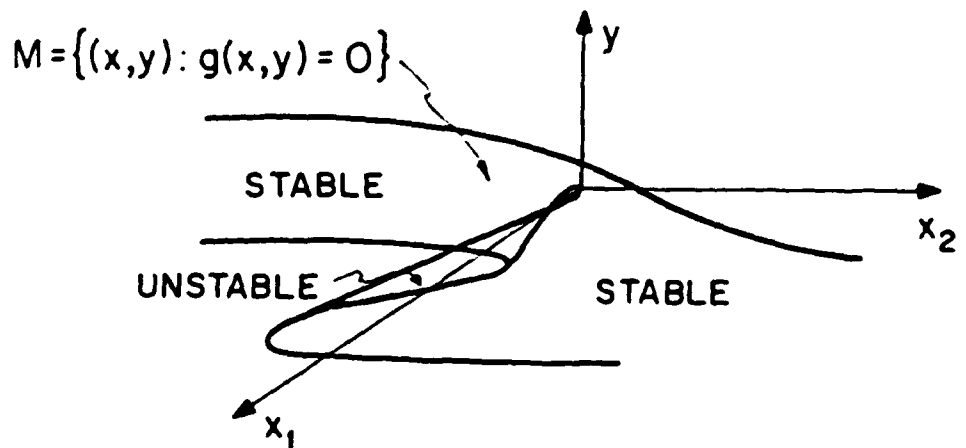
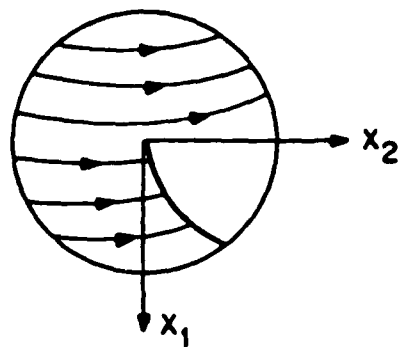
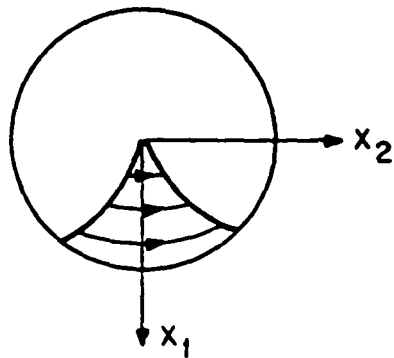


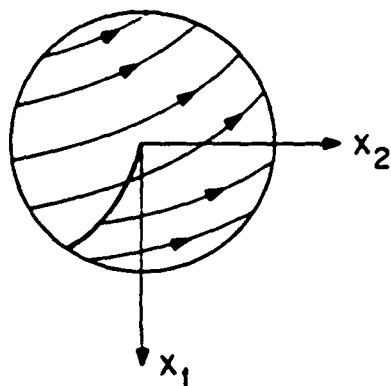
Figure 4: Visualization of a Cusp Bifurcation



UPPER SHEET



CENTER SHEET



BOTTOM SHEET

Figure 5: Visualization of the Trajectories on the  
Upper, Center and Bottom Sheets of the Cusp

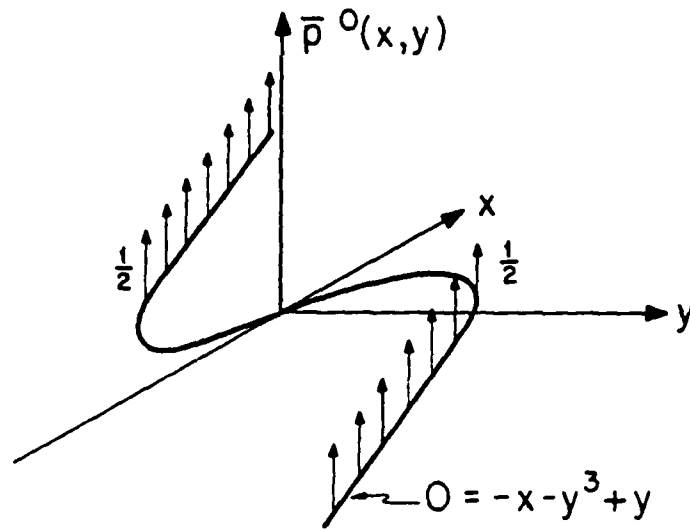


Figure 6: Showing the Limit as  $\lambda \downarrow 0$  of the Conditional Density  $\bar{p}^\lambda(x, y)$

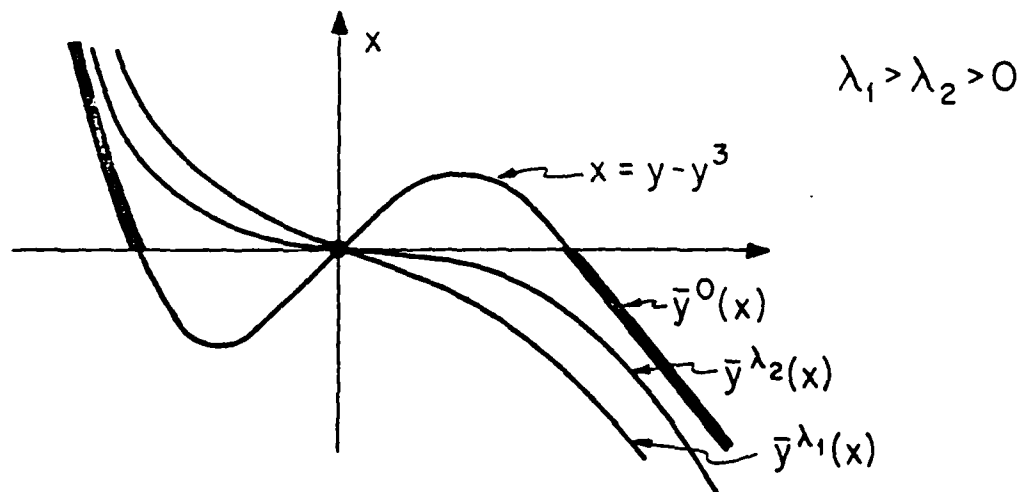


Figure 7: The Drift  $\bar{y}^\lambda(x)$  for the Limit Diffusion of the van der Pol Oscillator for Decreasing Values of  $\lambda$ .



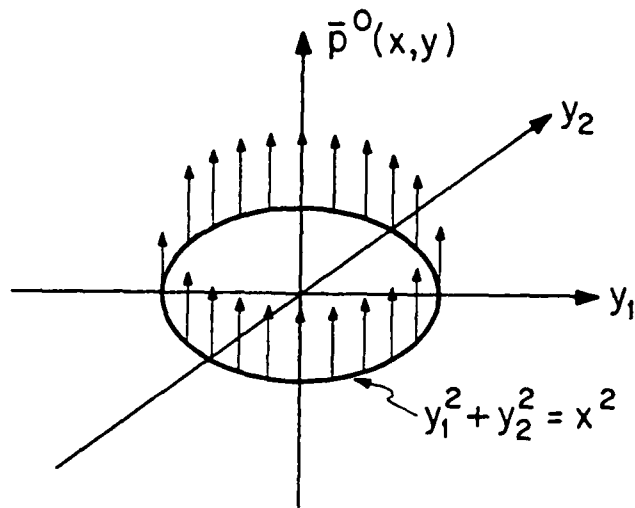


Figure 8: Showing the Limit Behavior of  $\bar{p}^\lambda(x,y)$  for Example 3.5

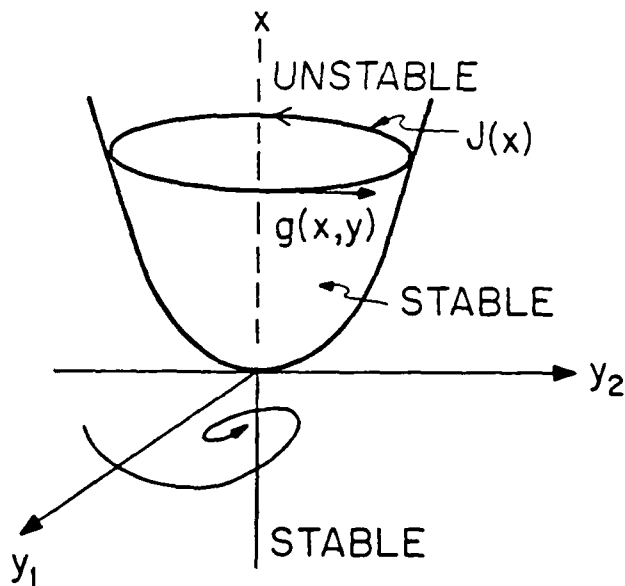


Figure 9: Showing the Hopf Bifurcation for the Example 3.6

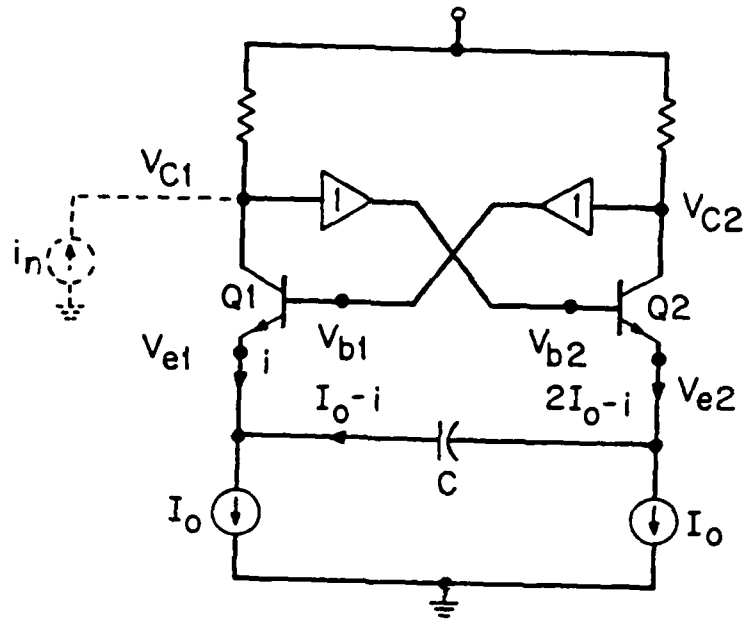


Figure 10: Simplified Circuit Diagram for the Emitter Coupled Relaxation Oscillator

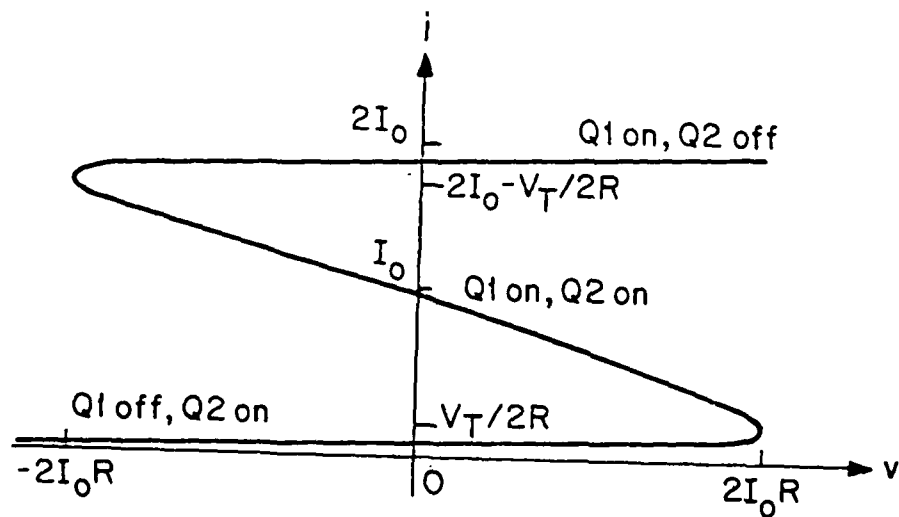


Figure 11: The Solution Curve to the Algebraic Equation (4.7)

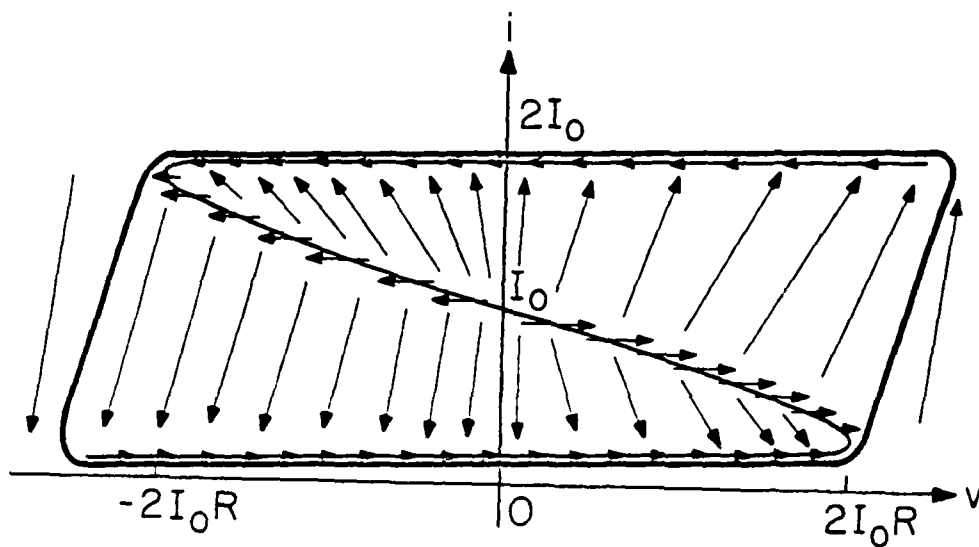


Figure 12: Phase Portrait of the System (4.6) , (4.9)

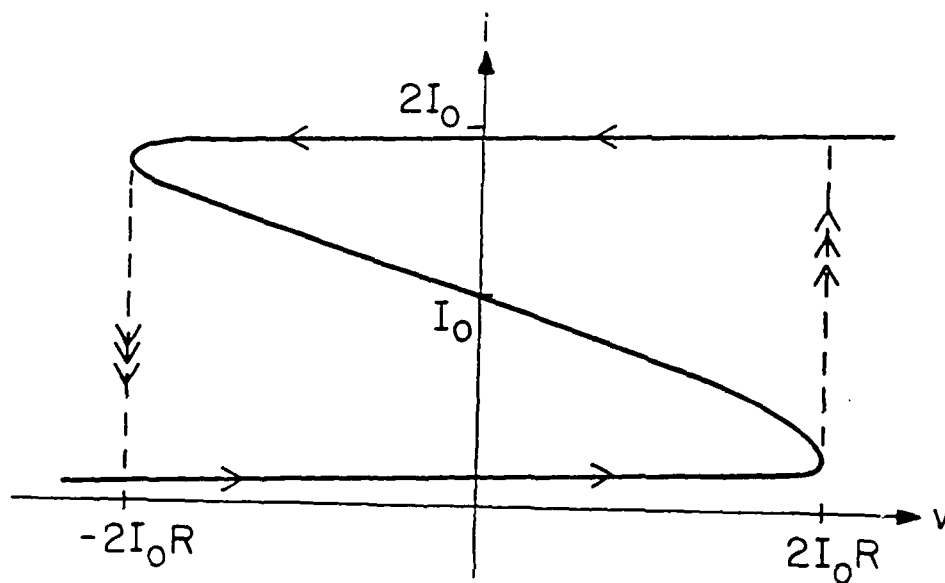


Figure 13: Showing the Relaxation Oscillation in the Emitter Coupled Oscillator

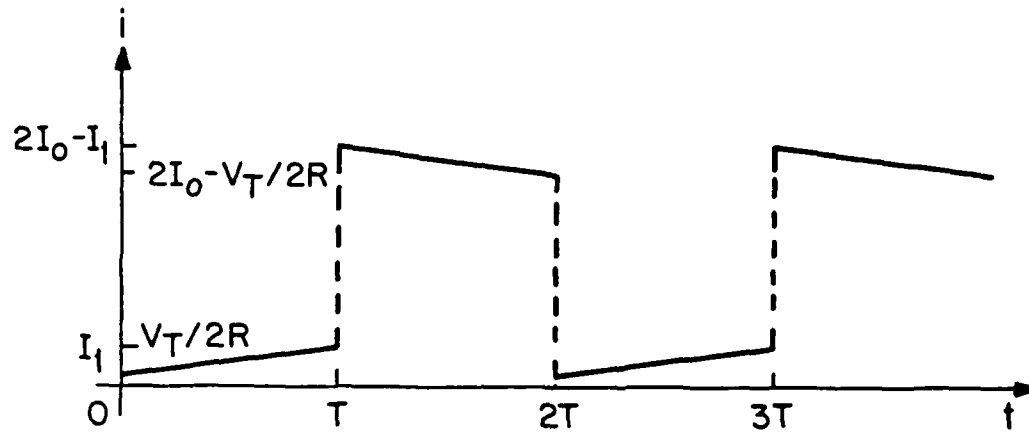


Figure 14: Current Waveform  $i(t)$  for the Circuit of Figure 1.

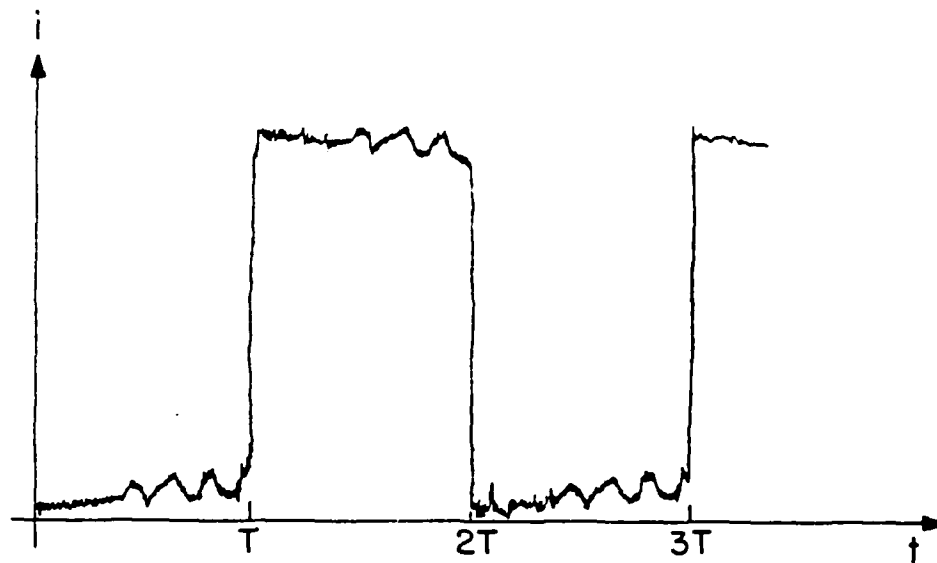


Figure 15: Experimentally Observed Waveform for  $i(t)$  in the Presence of Noise (after Abidi[1])

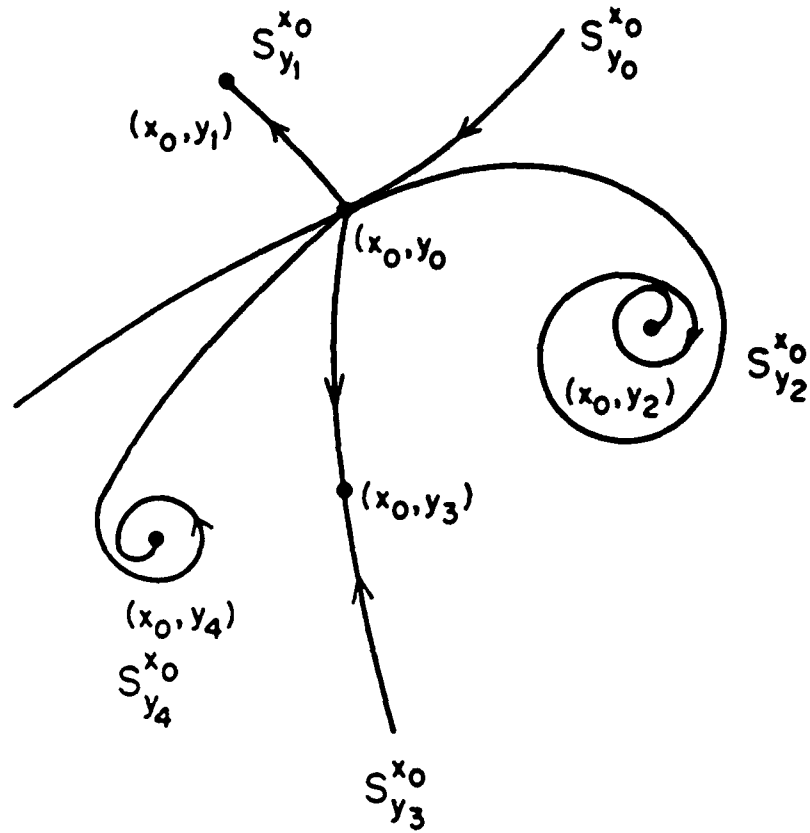


Figure 16: Showing the Possibility of going from  $y_1$  to  $y_2, y_3, y_4$  from  $y_0$

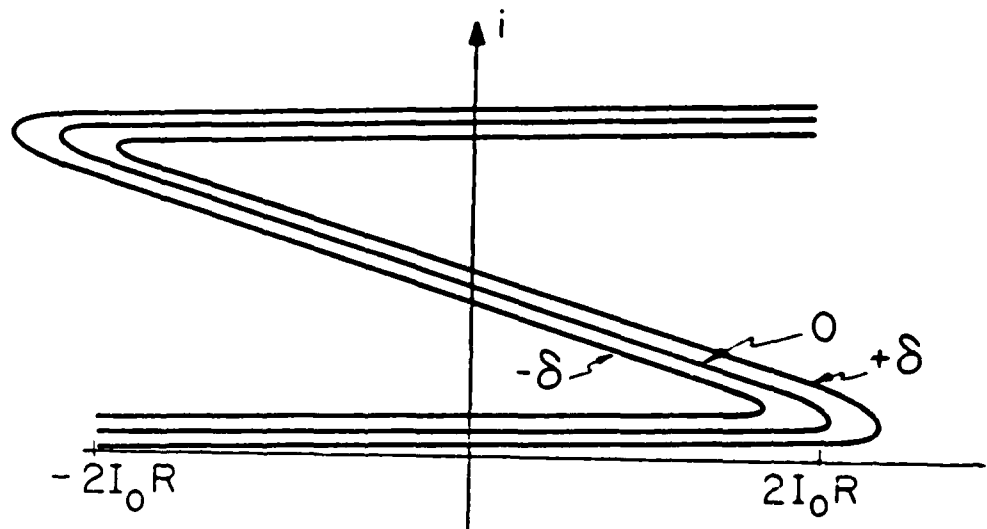


Figure 17: Showing a  $\delta$  neighbourhood of the Constraint Equation (4.7)

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